Coset Algebras of the Maxwell-Einstein Supergravities

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Abstract

The general structure of the scalar cosets of the Maxwell-Einstein supergravities is given. Following an introduction of the non-linear coset formalism of the supergravity theories a comparison of the coset algebras of the Maxwell-Einstein supergravities in various dimensions is discussed.

1 Introduction

The Maxwell-Einstein supergravities are obtained by coupling an arbitrary number of abelian vector multiplets to the supergravity multiplet in various dimensions. They may also be obtained by the Kaluza-Klein reduction on the Euclidean torus T^{10-D} of the ten dimensional simple $\mathcal{N}=1$ supergravity which is coupled to N abelian gauge multiplets [1]. The scalar cosets of the Maxwell-Einstein supergarvity theories can be formulated as non-linear sigma models, more specifically as the symmetric space sigma models. The study of the scalar cosets of these theories is essential in understanding the global symmetries of the entire theory. The global symmetry of the scalar lagrangian can be extended to the entire bosonic sector of the theory. The scalar cosets G/K of the Maxwell-Einstein theories are based on the global internal symmetry groups G which are in general non-compact real forms of semi-simple Lie groups [2]. Under certain conditions the global symmetry groups may be maximally non-compact (split) real forms but in general they are elements of a bigger class of Lie groups which contains the global symmetry groups of the maximal supergravities [3] namely the split real forms as a special subset [2, 4, 5, 6]. The main difference between the scalar cosets based on the non-compact and the maximally non-compact global symmetry groups is the parametrization that one can choose for the coset representatives. For the general non-compact real forms one can make use of the solvable Lie algebra gauge [7] to parameterize the scalar coset.

The Kaluza-Klein compactification of the bosonic sector of the ten dimensional simple $\mathcal{N}=1$ supergravity which is coupled to N abelian gauge multiplets [8] on the Euclidean torus T^{10-D} is given in [1]. When as a special case, the number of the $U(1)$ gauge fields is chosen to be 16, the ten dimensional supergravity which is coupled to 16 abelian $U(1)$ gauge multiplets becomes the low energy effective limit of the ten dimensional heterotic string theory. Thus the formulation in this case corresponds to the dimensional reduction of the low-energy effective bosonic lagrangian of the ten dimensional heterotic string theory. When the number of coupling vector multiplets is $N=16$, the $D=10$ Yang-Mills supergravity [8] has the $E_8 \times E_8$ Yang-Mills gauge symmetry, however the general Higgs vacuum structure causes a spontaneous symmetry breakdown so that the full symmetry $E_8 \times E_8$ is broken down to its maximal torus subgroup $U(1)^{16}$, whose Lie algebra is the Cartan subalgebra of $E_8 \times E_8$. Thus the ten dimensional Yang-Mills supergravity reduces to its maximal torus subtheory which is an abelian supergravity theory. The bosonic sector of this abelian Yang-Mills supergravity corresponds to the low energy effective Lagrangian of the bosonic sector of the fully Higgsed ten dimensional heterotic string theory [8]. This abelian supergravity theory is the one where the Maxwell-Einstein supergravities emerge from due to the Kaluza-Klein reduction as we have discussed above.

The method of non-linear realizations [9, 10, 11, 12, 13] is used in [14, 15] to formulate the gravity as a non-linear realization in which the gravity and the gauge fields appear on equal footing. Later, the dualization of the bosonic fields has provided the non-linear realization formulation of the bosonic sectors of the maximal supergravity theories [3, 4]. By introducing auxiliary fields for a subset of the field content and by using the coset formulation, the global symmetries of the scalar sectors of the maximal supergravities are studied in detail in [3]. These symmetries can also be realized on the bosonic fields. A general, dimension-independent formalism is developed for the bosonic sectors of the maximal supergravities in [16]. The coset realizations of the non-gravitational bosonic sectors of the $D = 11$ supergravity [17], the maximal supergravities which are obtained by the Kaluza-Klein reduction of the $D = 11$ supergravity over the tori T^n, as well as the IIB supergravity [18, 19, 20], are introduced by the dualization of the scalars and the higher-order gauge fields in [4]. In the same work, the twisted self-duality structure [21, 22] of the supergravities is generalized to regain the first-order equations of the corresponding theories from the Cartan forms of the dualized coset. Therefore in [4] it is shown that the non-linear coset formulation of the scalars can be improved to include the other non-gravitational bosonic fields, resulting in the first-order formulation of the relative theories. The mainline of [4] is to introduce dual fields for the non-gravitational bosonic fields and to construct the Lie superalgebra which will generate the coset representatives that realize the original field equations both in first and second-order by means of the Cartan form of the coset map. The dualization method is another manifestation of the lagrange multiplier methods which are used for the scalar sectors of the maximal supergravities in [3, 23].

In this work we discuss the relative structures of the coset algebras obtained as a result of the non-linear realization of the bosonic sectors of the Maxwell-Einstein supergarvities in $D = 7$ [24], $D = 8$ [25], $D = 9$ [26]. We will first mention about the general formulation of the scalar cosets of these theories in section two. In section three after briefly discussing the non-linear realization formalism we will give a comparison of the coset algebras of the Maxwell-Einstein supergravities which are constructed in [27, 28, 29] respectively.

2 Scalar Cosets

When the supergravity multiplet in D-dimensions is coupled to an arbitrary number of N abelian vector multiplets the scalars of the vector multiplets are governed by a symmetric space sigma model [3, 5, 6, 30, 31]. The scalar fields φ^{α} for $\alpha = 1, ..., N(10-D)$ parameterize the scalar coset manifold $SO(N, 10-D)/SO(N) \times SO(10-D)$ where $SO(N, 10-D)$ is in general a non-compact real form of a semi-simple Lie group and $SO(N) \times SO(10-D)$ is its maximal compact subgroup. For this reason $SO(N, 10-D)/SO(N) \times SO(10-D)$ is a Riemannian globally symmetric space for all the $SO(N, 10-D)$-invariant Riemannian structures on $SO(N, 10-D)/SO(N) \times SO(10-D)$ [2]. Therefore the scalar sector which consists of the vector multiplet scalars of the D dimensional Maxwell-Einstein supergravity can be formulated as a general symmetric space sigma model. To construct the symmetric space sigma model lagrangian one may make use of the solvable Lie algebra parametrization [7] for the parametrization of the scalar coset manifold $SO(N, 10-D)/SO(N) \times SO(10-D)$. The solvable Lie algebra parametrization is a consequence of the Iwasawa decomposition [2]

$$so(N,10-D) = \mathbf{k}_0 \oplus \mathbf{s}_0$$
$$= \mathbf{k}_0 \oplus \mathbf{h_k} \oplus \mathbf{n_k}, \tag{2.1}$$

where $\mathbf{k}_0$ is the Lie algebra of $SO(N) \times SO(10-D)$ and $\mathbf{s}_0$ is a solvable Lie subalgebra of $so(N,10-D)$. In (2.1) $\mathbf{h}_k$ is a subalgebra of the Cartan subalgebra $\mathbf{h}_0$ of $so(N,10-D)$ which generates the maximal R-split torus in $SO(N,10-D)$ [2, 6, 31]. The nilpotent Lie subalgebra $\mathbf{n_k}$ of $so(N,10-D)$ is generated by a subset $\{E_m\}$ of the positive root generators of $so(N,10-D)$ where $m \in \Delta^+_{nc}$. The roots in Δ^+_{nc} are the non-compact roots with respect to the Cartan involution θ induced by the Cartan decomposition [2, 31, 32]

$$so(N,10-D) = \mathbf{k}_0 \oplus \mathbf{u}_0, \tag{2.2}$$

where $\mathbf{u}_0$ is a vector subspace of $so(N,10-D)$. By using the scalar fields φ^α of the coupling vector multiplets and the generators of the solvable Lie algebra $\mathbf{s}_0$ we can parameterize the representatives of the scalar coset manifold $SO(N,10-D)/SO(N) \times SO(10-D)$ as [2]

$$L = exp(\frac{1}{2}\phi^i H_i)exp(\chi^m E_m), \tag{2.3}$$

where $\{H_i\}$ for $i = 1, ..., \dim(\mathbf{h}_k) \equiv r$ are the generators of $\mathbf{h}_k$ and $\{E_m\}$ for $m \in \Delta^+_{nc}$ are the positive root generators which generate $\mathbf{n_k}$. The scalars $\{\phi^i\}$ for $i = 1, ..., r$ are called the dilatons and $\{\chi^m\}$ for $m \in \Delta^+_{nc}$ are called the axions. The coset representatives satisfy the defining relation of $SO(N,10-D)$

$$L^T \eta L = \eta, \tag{2.4}$$

where $\eta = \mathrm{diag}(-,-,...,-,+,+,...,+)$ in which there are $10-D$ minus signs and N plus signs. If we assume that we choose the fundamental representation for the algebra $so(N, 10-D)$ and if we define the internal metric

$$\mathcal{M} = L^T L, \tag{2.5}$$

then the scalar lagrangian which governs the $N(10-D)$ scalar fields of the vector multiplets can be constructed as

$$\mathcal{L}_{scalar} = \frac{1}{4} tr(*d\mathcal{M}^{-1} \wedge d\mathcal{M}). \tag{2.6}$$

3 Coset Formulation and the Coset Algebras

In this section we will briefly mention about the general formalism which formulates the bosonic sectors of the Maxwell-Einstein supergravities as non-linear sigma models. As we have discussed before the coset construction of the bosonic sectors of the Maxwell-Einstein supergravities can be considered to be an extension of the coset structure of the scalars of these theories which we have given in the previous section. Such a formulation treats the scalars and the other bosonic fields on equal footing. One may apply the non-linear realization or the dualization method of [4] to construct the coset formulation of the bosonic sectors of the Maxwell-Einstein supergravity theories. Since the dualization method is another manifestation of the langrange multiplier methods the bosonic first-order formulation is also obtained as a consequence of the coset construction. In such a coset formulation one first defines a coset element which is generated by the bosonic fields coupled to the generators of a Lie superalgebra. Then the Lie superalgebra structure of the generators which parameterize this coset element is derived so that the Cartan form induced by the coset map realizes the field equations by satisfying the Cartan-Maurer equation. The Cartan form of the dualized coset element will obey a twisted self-duality equation [4, 33] which results in the first-order bosonic field equations of the theory. To construct the coset map the first task is to assign a generator for each bosonic field. The original generators $\{T_i\}$

are coupled to the fields $\{\tau^i\}$ in the coset parametrization. One should also introduce a dual field for each original field. The dual fields can be given as $\{\widetilde{\tau^i}\}$. These dual fields are the langrange multipliers coupling to the Bianchi identities of the field strengths of the original fields [23]. Thus if the original field is a p-form the dual field must be a $(D-p-2)-$form. We will also assign the dual generators $\{\widetilde{T}_i\}$ to the dual fields so that they will couple to the dual fields in the parametrization of the coset element. The Lie superalgebra of the original and the dual generators will have the Z_2 grading so that the generators will be odd if the corresponding potential is an odd degree differential form and otherwise even [4]. Specifically the doubled coset element will be parameterized by a differential graded algebra. This algebra is generated by the differential forms and the generators we have introduced above. The odd (even) generators behave like odd (even) degree differential forms under this graded differential algebra structure when they commute with the differential forms. The odd generators obey the anti-commutation relations while the even ones and the mixed ones obey the commutation relations.

Apart from the graviton e^r_μ the bosonic field content of the $D = 7, 8, 9$ dimensional Maxwell-Einstein supergravities can be given as

$$(B_{\mu\nu}, A^I_\mu, \sigma, \varphi^\alpha), \tag{3.1}$$

where $I = 1, ..., N + 10 - D$. The one-form fields A^I_μ include the N Maxwell fields of the vector multiplets and the $10-D$ vectors of the graviton multiplet. B is a two-form field of the graviton multiplet and σ is the dilaton of the graviton multiplet. The scalar fields φ^α belong to the vector multiplets as mentioned before. The construction of the coset formulation or the non-linear realization of the bosonic sector of the Maxwell-Einstein supergravities requires the definition of the coset element

$$\begin{aligned}\nu =& exp(\frac{1}{2}\phi^j H_j)exp(\chi^m E_m)exp(\sigma K)exp(A^I V_I)exp(\frac{1}{2}BY)\\ &\times exp(\frac{1}{2}\widetilde{B}\widetilde{Y})exp(\widetilde{A}^I\widetilde{V}_I)exp(\widetilde{\sigma}\widetilde{K})exp(\widetilde{\chi}^m\widetilde{E}_m)exp(\frac{1}{2}\widetilde{\phi}^j\widetilde{H}_j).\end{aligned} \tag{3.2}$$

Here we have defined the original generators $\{K, V_I, Y, H_j, E_m\}$ and as we have mentioned above the dual generators $\{\widetilde{K}, \widetilde{V}_I, \widetilde{Y}, \widetilde{H}_j, \widetilde{E}_m\}$. The coset map ν is a map from the D-dimensional spacetime into a group which is presumably the rigid symmetry group of the dualized lagrangian. However we will not focus on the group theoretical structure of the non-linear realization of the Maxwell-Einstein supergravities but rather on the Lie superalgebra which generates (3.2). Eventually this algebra also contains the information of the group theoretical structure of the coset formulation [2, 3, 5, 6, 30, 31]. The local map ν induces the Cartan form $\mathcal{G}$ on the D-dimensional spacetime which can be given as

$$\mathcal{G} = d\nu\nu^{-1}. \tag{3.3}$$

From its construction the Cartan form (3.3) satisfies the Cartan-Maurer equation

$$d\mathcal{G} - \mathcal{G}\wedge\mathcal{G} = 0. \tag{3.4}$$

The standard dualization procedure [4, 27, 28, 29, 30, 31] requires the construction of the Lie superalgebra of the original and the dual generators such that they will lead us to the second-order bosonic field equations of motion when the Cartan form (3.3) is calculated and inserted in (3.4). Therefore as performed in [27, 28, 29] one can calculate the Cartan form (3.3) in terms of the desired unknown structure constants of the Lie superalgebra of the original and the dual generators then one can insert this calculated Cartan form in the Cartan-Maurer equation (3.4) and finally compare the result with the second-order bosonic field equations to read the unknown structure constants. This is the general method to determine the Lie superalgebra structure which leads to the coset formulation of the Maxwell-Einstein supergravities. Next we will present and discuss the general structure of the coset algebras obtained as a result of the above mentioned coset formulation of the $D=7$ [24], $D=8$ [25], $D=9$ [26] Maxwell-Einstein supergravities.

3.1 The D=7 Case

The bosonic lagrangian of the $\mathcal{N}=2$, $D=7$ Maxwell-Einstein supergravity can be given as [24]

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}R*1 - \frac{5}{8}*d\sigma\wedge d\sigma - \frac{1}{2}e^{2\sigma}*G\wedge G \\ &\quad - \frac{1}{8}tr(*d\mathcal{M}^{-1}\wedge d\mathcal{M}) - \frac{1}{2}e^{\sigma}F\wedge\mathcal{M}*F,\end{aligned} \tag{3.5}$$

where the coupling between the field strengths $F^I = dA^I$ for $I = 1, ..., N+3$ and the scalars which parameterize the coset $SO(N,3)/SO(N)\times SO(3)$ can be explicitly written as

$$-\frac{1}{2}e^{\sigma}F\wedge\mathcal{M}*F = -\frac{1}{2}e^{\sigma}\mathcal{M}_{ij}F^i\wedge *F^j. \tag{3.6}$$

We have assumed the $(N+3)$-dimensional matrix representation of $so(N,3)$. We have $\eta = \text{diag}(-,-,-,+,+,...,+)$. The Chern-Simons form G is defined as [24]

$$G = dB - \frac{1}{\sqrt{2}}\,\eta_{ij}\,A^i\wedge F^j. \tag{3.7}$$

In [27] the method of dualization which we have introduced above is applied for the lagrangian (3.5) and the coset algebra of the $\mathcal{N}=2$, $D=7$ Maxwell-Einstein supergravity is found to be

$$[K, V_i] = \frac{1}{2}V_i, \qquad [K, Y] = Y, \qquad [K, \widetilde{Y}] = -\widetilde{Y},$$

$$[\widetilde{V}_k, K] = \frac{1}{2}\widetilde{V}_k, \qquad \{V_i, V_j\} = -\frac{1}{\sqrt{2}}\eta_{ij}Y, \qquad [H_l, V_i] = (H_l)_i^k V_k,$$

$$[E_m, V_i] = (E_m)_i^j V_j, \qquad [V_l, \widetilde{V}_k] = -\frac{2}{5}\delta_{lk}\widetilde{K} + \frac{1}{2}\sum_{i=1}^{r}(H_i)_{lk}\widetilde{H}_i,$$

$$\{V_k, \widetilde{Y}\} = 2\sqrt{2}\,\eta_k^l\,\widetilde{V}_l, \qquad [Y, \widetilde{Y}] = \frac{16}{5}\widetilde{K}, \qquad [H_i, \widetilde{V}_k] = -(H_i^T)_k^m\widetilde{V}_m,$$

$$[E_\alpha, \widetilde{V}_k] = -(E_\alpha^T)_k^m\widetilde{V}_m, \qquad [H_j, E_\alpha] = \alpha_j E_\alpha, \qquad [E_\alpha, E_\beta] = N_{\alpha,\beta}E_{\alpha+\beta},$$

$$[H_j, \widetilde{E}_\alpha] = -\alpha_j\widetilde{E}_\alpha, \qquad [E_\alpha, \widetilde{E}_\alpha] = \frac{1}{4}\sum_{j=1}^{r}\alpha_j\widetilde{H}_j,$$

$$[E_\alpha, \widetilde{E}_\beta] = N_{\alpha,-\beta}\widetilde{E}_\gamma, \qquad \alpha - \beta = -\gamma,\ \alpha \neq \beta, \tag{3.8}$$

where we have also included the commutation relations of the generators of the solvable Lie subalgebra $\mathbf{s}_0$ of $so(N,3)$ which form up a subalgebra in (3.8). The matrices $((H_m)_i^j,\ (E_\alpha)_i^j)$ are the matrix representatives of the corresponding generators (H_m, E_α). Also the matrices $((H_m^T)_i^j,\ (E_\alpha^T)_i^j)$ are the matrix transpose of $((H_m)_i^j,\ (E_\alpha)_i^j)$. The commutators and the anti-commutators which are not listed in (3.8) vanish.

3.2 The D=8 Case

The lagrangian of the bosonic sector of the $\mathcal{N}=1$, $D=8$ Maxwell-Einstein supergravity is given as [25]

$$\begin{aligned}\mathcal{L} &= \frac{1}{4}R*1+\frac{3}{8}d\sigma\wedge *d\sigma-\frac{1}{2}e^{2\sigma}G\wedge *G \\ &+\frac{1}{16}tr(d\mathcal{M}^{-1}\wedge *d\mathcal{M})-\frac{1}{2}e^{\sigma}F\wedge\mathcal{M}*F.\end{aligned} \tag{3.9}$$

The Chern-Simons three-form G is

$$G=dB+\eta_{ij}F^i\wedge A^j. \tag{3.10}$$

The scalars parameterize the coset manifold $SO(N,2)/SO(N)\times SO(2)$ and the $SO(N,2)$ invariant tensor η is $\eta=\mathrm{diag}(-,-,+,+,...,+)$ as generally defined before. We assume that we choose an $(N+2)$-dimensional matrix representation of $so(N,2)$. In [28] the coset algebra which leads to the non-linear sigma model or the coset formulation of the bosonic sector of the $\mathcal{N}=1$, $D=8$ Maxwell-Einstein supergravity is derived as

$$[K,V_i]=\frac{1}{2}V_i,\qquad [K,Y]=Y,\qquad [K,\widetilde{Y}]=-\widetilde{Y},$$

$$[\widetilde{V}_k,K]=\frac{1}{2}\widetilde{V}_k,\qquad \{V_i,V_j\}=\eta_{ij}Y,\qquad [H_l,V_i]=(H_l)^k_iV_k,$$

$$[E_m,V_i]=(E_m)^j_iV_j,\qquad \{V_l,\widetilde{V}_k\}=\frac{2}{3}\delta_{lk}\widetilde{K}+\sum_{i=1}^{r}(H_i)_{lk}\widetilde{H}_i,$$

$$[V_k,\widetilde{Y}]=-4\,\eta^l_k\,\widetilde{V}_l,\qquad [Y,\widetilde{Y}]=-\frac{16}{3}\widetilde{K},\qquad [H_i,\widetilde{V}_k]=-(H_i^T)^m_k\widetilde{V}_m,$$

$$[E_\alpha, \widetilde{V}_k] = -(E_\alpha^T)_k^m \widetilde{V}_m, \qquad [H_j, E_\alpha] = \alpha_j E_\alpha, \qquad [E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta},$$

$$[H_j, \widetilde{E}_\alpha] = -\alpha_j \widetilde{E}_\alpha, \qquad [E_\alpha, \widetilde{E}_\alpha] = \frac{1}{4}\sum_{j=1}^{r} \alpha_j \widetilde{H}_j,$$

$$[E_\alpha, \widetilde{E}_\beta] = N_{\alpha,-\beta} \widetilde{E}_\gamma, \qquad \alpha - \beta = -\gamma, \alpha \neq \beta. \tag{3.11}$$

As in the $D = 7$ case the matrices $((H_m)_i^j,\ (E_\alpha)_i^j)$ are the matrix representatives of the corresponding generators (H_m, E_α) in the $(N+2)$-dimensional matrix representation of $so(N,2)$. Also the matrices $((H_m^T)_i^j,\ (E_\alpha^T)_i^j)$ are the matrix transpose of $((H_m)_i^j,\ (E_\alpha)_i^j)$. The commutators and the anti-commutators which are not listed in (3.11) vanish.

3.3 The D=9 Case

Finally we will present the coset algebra of the $\mathcal{N} = 1$, $D = 9$ Maxwell-Einstein supergravity [26] which is derived in [29]. The bosonic lagrangian of the $\mathcal{N} = 1$, $D = 9$ Maxwell-Einstein supergravity can be given as [26]

$$\begin{aligned} \mathcal{L} = &-\frac{1}{4}R * 1 + \frac{7}{4} * d\sigma \wedge d\sigma + \frac{1}{2}e^{-4\sigma} * G \wedge G \\ &+ \frac{1}{16}tr(*d\mathcal{M}^{-1} \wedge d\mathcal{M}) - \frac{1}{2}e^{-2\sigma} F \wedge \mathcal{M} * F. \end{aligned} \tag{3.12}$$

In this case the scalars of the coupling abelian vector multiplets parameterize the scalar coset $SO(N,1)/SO(N)$. The Chern-Simons three-form is taken as

$$G = dB + \eta_{IJ} A^I \wedge F^J. \tag{3.13}$$

We assume an $(N+1)$-dimensional matrix representation of $so(N,1)$. As before we have $\eta = \mathrm{diag}(-,+,+,...,+)$. The coset algebra which parameterizes the coset (3.2) and which generates the coset formulation of the bosonic sector is derived in [29] as

$$[K,V_I] = -V_I, \qquad [K,Y] = -2Y, \qquad [K,\widetilde{Y}] = 2\widetilde{Y},$$

$$[\widetilde{V}_I,K] = -\widetilde{V}_I, \qquad \{V_I,V_J\} = \eta_{IJ}Y, \qquad [H_l,V_I] = (H_l)_I^K V_K,$$

$$[E_m,V_I] = (E_m)_I^J V_J, \qquad [V_L,\widetilde{V}_M] = -\frac{2}{7}\delta_{LM}\widetilde{K} - \sum_{i=1}^{r}(H_i)_{LM}\widetilde{H}_i,$$

$$\{V_K,\widetilde{Y}\} = 4\eta_K^L\widetilde{V}_L, \qquad [Y,\widetilde{Y}] = -\frac{16}{7}\widetilde{K}, \qquad [H_i,\widetilde{V}_K] = -(H_i^T)_K^M\widetilde{V}_M,$$

$$[E_\alpha,\widetilde{V}_K] = -(E_\alpha^T)_K^M\widetilde{V}_M, \qquad [H_j,E_\alpha] = \alpha_j E_\alpha, \qquad [E_\alpha,E_\beta] = N_{\alpha,\beta}E_{\alpha+\beta},$$

$$[H_j,\widetilde{E}_\alpha] = -\alpha_j\widetilde{E}_\alpha, \qquad [E_\alpha,\widetilde{E}_\alpha] = \frac{1}{4}\sum_{j=1}^{r}\alpha_j\widetilde{H}_j,$$

$$[E_\alpha,\widetilde{E}_\beta] = N_{\alpha,-\beta}\widetilde{E}_\gamma, \qquad \alpha-\beta=-\gamma,\ \alpha\neq\beta. \tag{3.14}$$

As in the other cases the commutators and the anti-commutators which are not listed above vanish.

We know that in general for the global symmetry algebra $so(N,10-D)$ in the D-dimensional Maxwell-Einstein supergravity the dimension of the solvable Lie algebra $\mathbf{s}_0$ is $N(10-D)$. Thus in a D-dimensional theory we have $N(10-D)$ dilatons and axions as the scalars of the vector multiplets which parameterize the scalar coset $SO(N,10-D)/SO(N)\times SO(10-D)$. From the coset algebras (3.8), (3.11), (3.14) we also observe that these algebras contain the solvable Lie algebras $\mathbf{s}_0$ in them. One more observation is that the scalar field generators and their duals form up a subalgebra in each case. Thus one may think of the coset algebras as extensions of the $2N(10-D)$ dimensional scalar-dual algebras in each case whereas the scalar-dual algebras can be considered to be the extensions of the solvable Lie algebras $\mathbf{s}_0$ which parameterize the scalar coset manifolds $SO(N,10-D)/SO(N)\times SO(10-D)$. When we consider the bosonic field content of the D-dimensional Maxwell-Einstein supergravities in general we have a supergravity multiplet dilaton σ, a two form field B, $N(10-D)$ vector multiplet scalars and we have $N+(10-D)$ one-form fields. For the coset construction of the bosonic sector one has to double the field content by introducing dual fields. Since we also assign a generator for each field the dimension of the coset algebra becomes

$$\dim(s_{dual}) = 22N-(2N+2)D+24. \tag{3.15}$$

We deduce that the general scheme of the coset algebras in various dimensions is the same however as a result of the coset construction method we have discussed before, due to the oddness or the evenness of the dimension D of the spacetime the odd-even structure of the generators differ. For all the coset algebras given in (3.8), (3.11) and (3.14) the generators $\{\widetilde{K},\widetilde{H}_i\}$ commute with all the algebra generators thus they generate the center $(s_{dual})_c$ of the coset algebras s_{dual} in each dimension. Therefore we have

$$\dim((s_{dual})_c) = \dim(\mathbf{h}_k)+1. \tag{3.16}$$

Finally we find that there are several abelian subalgebras of the coset algebras s_{dual} which are generated by the sets

$$\{\widetilde{K},\widetilde{V}_I,\widetilde{Y},\widetilde{H}_j,\widetilde{E}_m\},\quad \{\widetilde{K},\widetilde{V}_I,Y,\widetilde{H}_j,\widetilde{E}_m\},\quad \{\widetilde{K},Y,\widetilde{H}_j,H_j\},$$

$$\{\widetilde{K}, K, \widetilde{H}_j, \widetilde{E}_m\}, \quad \{\widetilde{K}, K, \widetilde{H}_j, H_j\}, \quad \{\widetilde{K}, H_i, \widetilde{Y}, \widetilde{H}_j\},$$

$$\{\widetilde{K}, H_i, Y, \widetilde{H}_j\}. \tag{3.17}$$

The dimension of the first two of these algebras which are maximal in dimension is

$$11N - (N+1)D + 12, \tag{3.18}$$

which is half of the dimension of the coset algebras s_{dual}. The dimension of the abelian subalgebra generated by $\{\widetilde{K}, K, \widetilde{H}_j, \widetilde{E}_m\}$ is

$$N(10 - D) + 2, \tag{3.19}$$

and the dimension of the rest of the abelian subalgebras of s_{dual} which are given in (3.17) is

$$2\,\dim\mathbf{h_k} + 2. \tag{3.20}$$

As a final remark we can state that if the coset algebras (3.8), (3.11), (3.14) form up a solvable Lie algebra parametrization for a dualized coset structure G_{dual}/K_{dual} then the first two of the above mentioned abelian algebras in (3.17) can be the candidates for the subalgebra of the Cartan subalgebra of g_{dual} which generates the maximal R-split torus in G_{dual}.

4 Conclusion

In section two we have discussed the scalar coset structures of the Maxwell-Einstein supergravities in general. After mentioning the general formalism of the non-linear sigma model or the coset formulation of the bosonic sectors of the Maxwell-Einstein supergravity theories we have given a comparison of the coset algebras derived in the coset constructions in various dimensions in section three.

The symmetries of the supergravity theories have been studied in the recent years to gain insight in the symmetries and the duality transformations of the string theories whose low energy effective limits or the massless sectors are the supergravities. The global symmetries of the supergravities help us to understand the non-perturbative U-duality symmetries of the string theories and the M theory [34, 35]. A restriction of the global symmetry group G of the supergravity theory to the integers Z, namely $G(Z)$, is conjectured to be the U-duality symmetry of the relative string theory which unifies the T-duality and the S-duality [34]. Therefore the coset formulation of the supergravities have not only enabled us to study the symmetries of the supergravities in detail but also provided a better understanding of the dualities and the symmetries of the string theories whose low energy effective limits are the relative supergravities.

The Lie superalgebras we have presented in section three generate the dualized coset elements. They may be considered as the parametrization of a coset structure G_{dual}/K_{dual} of the bosonic sector likewise the coset structure G/K of the scalars. We know that the global (rigid) symmetries of the scalar sector whose action on the scalar fields does not depend on the spacetime coordinates are essential to have a deeper understanding of the supergravity theories. One can also define the action of the global symmetry group of the scalars on the other fields as well, thus the global symmetry of the scalars can be extended to be the global symmetry of not only the bosonic sector but the entire theory. The groups of the coset formulations namely G_{dual} in various dimensions can be studied as enlarged global symmetry groups of the corresponding Maxwell-Einstein supergravity theories. Therefore as we have mentioned before the improved global symmetry analysis of the Maxwell-Einstein supergravities is an essential tool to study the symmetry scheme of the relative heterotic string theories since as discussed in the introduction the Maxwell-Einstein supergravities are the low energy limits of the heterotic string theories. In particular the T^3-compactification of the $D = 10$ type I supergravity that is coupled to the Yang-Mills theory [8, 36] which forms up the low energy effective limit of the $D = 10$ heterotic string gives the $D = 7$ Maxwell-Einstein supergravity. Also an equivalent dual bosonic lagrangian of the $D = 7$ Maxwell-Einstein supergravity in which the two-form potential B is replaced by a dual three-form field [37, 38] corresponds to the K_3-compactification [39] of the $D = 11$ supergravity [17] which is conjectured to be the low energy limit of the M theory. Thus by constructing the coset algebra which reveals information about the global symmetries of the $D = 7$ Maxwell-Einstein supergravity we may have insight into the symmetries of M theory. In [40] the string-membrane dualities in $D = 7$ which arise from the comparison of the construction of the $D = 7$ Maxwell-Einstein supergravity either as a toroidally compactified heterotic string or a K_3-compactified $D = 11$ supermembrane are discussed. The coset formulation of the bosonic sector of the $D = 7$ Maxwell-Einstein supergravity will also help to understand the string-membrane and the string-string dualities in $D = 7$.

The identification of the coset algebras we have given in section three is not done. As a first guess one may consider them as solvable algebras which are parts of Iwasawa decompositions similar to the algebraic construction of the scalar cosets we have mentioned in section two. For this reason one may focus on the identification of the abelian and the nilpotent parts of s_{dual} by considering necessary generator redefinitions. The relation of these algebras can be inspected with the coset algebras of the $D = 11$ and the maximal supergravities [4, 41, 42] as well. The attempt to identify the coset structure G_{dual}/K_{dual} and thus the enlarged global symmetry group G_{dual} must be in correspondence with the identification of the coset algebras. However we should point out that likewise proposed and applied in [3] we make use of a differential graded algebra of the module of the differential forms and the algebra of the field generators for the parametrization of the coset therefore the group theoretical considerations of the coset formulation should posses structures more involved than Lie groups. If one manages to construct the group theoretical framework of the coset formulation one would obtain a legitimate geometrical formulation of the related supergravity theory at least for the bosonic sector. Another essential inquiry may be held for the realization of the action of the proposed global symmetry group G_{dual} on the fields. In [4] it is argued that the symmetry groups generated by the coset algebras become the symmetry groups of the Cartan forms induced by the coset maps. Also they are conjectured to be the symmetry groups of the first-order field equations obtained by a twisted self-duality condition satisfied by the Cartan forms. One may advance in this direction not only to identify the enlarged global symmetry group G_{dual} but also to explore the transformation laws of its action on the field contents of the related supergravities. If one manages to find out the nature of G_{dual} one may also work on the construction of a dualized lagrangian which includes the original and the dual fields and work out the action of G_{dual} on the dualized lagrangian.

In [43] the method of non-linear realizations is used to derive the dynamics of the M theory branes. Furthermore the M theory branes when they are in a background are also described as non-linear realizations in [41]. Thus as a final remark the comparison of the coset structures obtained in these works with the coset algebras we have studied may reveal new facts about the brane dynamics.

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